the case $a_1 = 3$, $b_1 = 1$, $\omega_1 = 0$, $a_2 = 2$, $b_2 = 1$, $\omega_2 = 30$ deg. The data for these are set out as follows:

$$a = 2.38929, \quad b = 1.37061, \quad \omega = 24.048 \text{ deg}$$

$$\Theta_1 = 61.245 \text{ deg}, \quad \Theta_2 = 185.085 \text{ deg}$$

$$u_1 = 3.48106, \quad u_2 = 1.09306$$

$$\phi_1 = 7.038 \text{ deg}, \quad \phi_2 = 8.425 \text{ deg}$$

$$A_1 = -2.09028, \quad A_2 = -1.23814$$

$$V = 0.31058\gamma^{1/2}$$

$$a = 2.51336, \quad b = 0.53909, \quad \omega = 12.244 \text{ deg}$$

$$\Theta_1 = 164.989 \text{ deg}, \quad \Theta_2 = 46.883 \text{ deg}$$

$$u_1 = 2.03413, \quad u_2 = 2.95690$$

$$\phi_1 = 3.257 \text{ deg}, \quad \phi_2 = 3.062 \text{ deg}$$

$$A_1 = -1.45305, \quad A_2 = -1.74807$$

$$V = 0.33488\gamma^{1/2}$$

Both solutions provide a local minimum for V, but the first is believed to represent the global optimal solution.

It appears, therefore, that in this case there are four minima for V and two stationary values (at least).

Conclusions

It has been shown that the equations determining the optimal two-impulse mode of transfer between coplanar elliptical orbits have many solutions, some of which yield values for the characteristic velocity of the maneuver that are local minima. These include special solutions that are readily derived from the equations and other solutions that can only be reached by use of an iterative computer program; the global optimal transfer is generally of the latter type. Solutions for which the transfer orbit is circular generate values of the characteristic velocity that are locally stationary but are not minima.

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Determining the Largest Hypersphere of Stability Using Lagrange Multipliers

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Introduction

ONE of the most important characteristics of a good control system is its ability to remain stable for uncertain but bounded parameter variations. This is referred to as stability robustness. Indeed, a primary reason for utilizing feedback control instead of open-loop control is the presence of model uncertainties. Uncertainties appear in the system mathematical model due to many factors, e.g., component tolerances, measurement errors, and linearization approximation. For example, if a mathematical model is developed for a system whose elements are known within a certain amount of tolerance, then the resulting system matrix can have entries that vary within some fixed bounds and around some nominal values. Thus, it is an important problem to obtain a quantitative scalar measure of allowable perturbations that can be tolerated by a system without affecting its stability.

Consider the state space model of a linear system:

$$\dot{x}(t) = (A^0 + E)x(t) = Ax(t) \tag{1}$$

where A^0 is the nominal system matrix and E is the perturbation matrix with a known structure. The perturbation as characterized by E is classified further into highly structured perturbation, in which the bounds on individual elements of the perturbation matrix are known, and weakly structured perturbation in which a bound on some norm of the perturbation matrix is known with no knowledge about the bounds of individual elements.

Robust stability criteria developed in the time domain are primarily based on the Lyapunov theory.[1-3] The results obtained carry some conservativeness for determining the upper bounds of the individual parameters as well as for finding the spectral norm bound of the perturbation matrix.[4]

The perturbation bound that we shall introduce next, namely, the radius of the largest hypersphere of stability, gives the designer the ability to obtain the maximum allowable Euclidean norm on the perturbation matrix elements under weakly structured perturbation as well as bounds on individual elements under highly structured perturbations. This will be accomplished by finding the boundary hypersurfaces that separate the parameter space into stable and unstable regions based on the results of Jury and Pavilidis.[5] By finding the shortest distance from the stable nominal point to the respective boundary hypersurfaces, the largest hypersphere centered at the nominal point is found in which elements of the system matrix can vary without affecting system stability. The effectiveness of the method will be illustrated by obtaining closed-form solutions for second-order systems and an example for a third-order system.

Main Results

Starting with the system description given in Eq. (1), the characteristic polynomial can be found by

$$f(s) = \det[sI - A] = b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0 \tag{2}$$

The coefficients b_i are functions of the elements of A. Based on Jury and Pavlidis' result,[5] the critical conditions for stability limits are given by

$$b_0 = 0 \tag{3a}$$

and

$$\Delta_{n-1} = \begin{bmatrix} b_{n-1} & b_{n-3} & b_{n-5} & b_{n-7} & \cdot & 0 \\ b_n & b_{n-2} & b_{n-4} & b_{n-6} & \cdot & \cdot \\ 0 & b_{n-1} & b_{n-3} & b_{n-5} & \cdot & \cdot \\ 0 & b_n & b_{n-2} & b_{n-4} & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdot & b_0 \\ 0 & 0 & 0 & 0 & \cdot & b_1 \end{bmatrix} = 0$$

$$n > 1 \tag{3b}$$

where $\Delta_{n-1} \epsilon R^{(n-1)\times(n-1)}$. These two critical conditions for stability limits separate the parameter space into stable and

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unstable regions. By finding the shortest distance from the stable nominal point to the respective boundary hypersurfaces, the largest hypersphere centered at the nominal point is found in which elements of the system matrix can vary without affecting system stability. The two critical conditions for stability limits in Eq. (3) can be formulated as two boundary hypersurfaces in the parameter space as

$$b_0 \equiv h_{s1}(a_{11}, a_{12}, \ldots, a_{nn}) = 0 \tag{4a}$$

$$\Delta_{n-1} \equiv h_{s2}(a_{11}, a_{12}, \ldots, a_{nn}) = 0 \tag{4b}$$

The squared distance R_s^2 from the nominal point to any point on a boundary hypersurface is given by

$$R_s^2(a_{11}, a_{12}, \ldots, a_{nn}) = (a_{11} - a_{11}^0)^2 + (a_{12} - a_{12}^0)^2 + \cdots + (a_{nn} - a_{nn}^0)^2 \tag{5}$$

The function described in Eq. (5) can be minimized under the constraints of Eq. (4) by using the Lagrange multiplier approach. Let

$$W_{s1}(a_{11}, a_{12}, \ldots, a_{nn}, \lambda_{s1}) = R_s^2 + \lambda_{s1} h_{s1} \tag{6a}$$

$$W_{s2}(a_{11}, a_{12}, \ldots, a_{nn}, \lambda_{s2}) = R_s^2 + \lambda_{s2} h_{s2} \tag{6b}$$

where λ_{s1} and λ_{s2} are the Lagrange multipliers. The points on the first and second boundary surfaces that are closest to the nominal point must satisfy the following two sets of equations, respectively:

$$\frac{\partial W_{s1}}{\partial a_{11}} = 0, \quad \frac{\partial W_{s1}}{\partial a_{12}} = 0, \quad \ldots, \quad \frac{\partial W_{s1}}{\partial a_{nn}} = 0, \quad \frac{\partial W_{s1}}{\partial \lambda_{s1}} = 0 \tag{7a}$$

$$\frac{\partial W_{s2}}{\partial a_{11}} = 0, \quad \frac{\partial W_{s2}}{\partial a_{12}} = 0, \quad \ldots, \quad \frac{\partial W_{s2}}{\partial a_{nn}} = 0, \quad \frac{\partial W_{s2}}{\partial \lambda_{s2}} = 0 \tag{7b}$$

Solving for *all* roots of Eqs. (7a) and (7b), respectively, and examining these in turn will yield the minimum squared distances $W_{s1,\min}$ and $W_{s2,\min}$ from the nominal point to the first and second boundary hypersurfaces. It follows that the squared radius of the largest hypersphere of all stable points for A is given by

$$R_{s,\max}^2 = \min[W_{s1,\min}, \; W_{s2,\min}] \tag{8}$$

which is the maximum allowable bound under weakly structured perturbations. It also means that the elements of A can be perturbed without affecting stability as long as

$$R_{s,\max}^2 > \Delta a_{11}^2 + \Delta a_{12}^2 + \cdots + \Delta a_{nn}^2 \tag{9}$$

which could be used to obtain bounds on $a_{11}, a_{12}, \ldots, a_{nn}$, i.e., bounds on the system matrix parameters under highly structured perturbations.

Finding *all* roots of a system of multidimensional polynomials as in Eq. (7) can be accomplished numerically using homotopy or continuation methods.[6-9] Also, software for solving such a system of equations is available, e.g., Mathematica.

For second-order systems, given the nominally stable system matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{10}$$

the characteristic polynomial is

$$f(s) = s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) \tag{11}$$

Using the approach described previously, the maximum radius of stability for all possible combinations of perturbation in the elements of the system matrix is given in Table 1.

Examples

Example 1

Consider a second-order system matrix used by Yedavalli[3]:

$$A^0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

Take the case when all of the elements of the system matrix are subject to perturbation. From the last row of Table 1, with $a_{11}^0 = -3$, $a_{12}^0 = -2$, $a_{21}^0 = 1$, $a_{22}^0 = 0$, R_s^2 is calculated to be 0.54. Hence, the stability condition under weakly structured perturbation is

$$(\Delta a_{11})^2 + (\Delta a_{12})^2 + (\Delta a_{21})^2 + (\Delta a_{22})^2 < 0.54 \tag{12}$$

For the sake of comparison with Yedavalli's result, the magnitudes of the maximum deviations for each element with the assumption that all elements can be perturbed to the same maximum amount, i.e., $|\Delta a_{11}| = |\Delta a_{12}| = |\Delta a_{21}| = |\Delta a_{22}|$, is found from Eq. (12) to be 0.367. The results for different combinations of perturbing elements are calculated in the same manner and are given by the third column in Table 2. For comparison, the magnitudes of the maximum deviations for each element as computed in Yedavalli's paper[3] are reproduced here by the second column in Table 2. It is noted that in all cases Yedavalli's bounds μ_y are contained within the bounds μ_p of the presented method.

Example 2

Consider the third-order system matrix used by Ref. 10 constructing the parameter stability region:

$$A = \begin{bmatrix} -1 & a_{12} & 3 \\ 1 & a_{22} & 1 \\ -5 & 1 & -2 \end{bmatrix}$$

where $a_{12}^0 = 2$ and $a_{22}^0 = -3$. From Eqs. (3) and (4), the equations for the two boundary hypersurfaces are

$$h_{s1}(a_{12}, a_{22}) = -4 + 3a_{12} - 17a_{22} = 0 \tag{13a}$$

$$h_{s2}(a_{12}, a_{22}) = 52 - 6a_{12} + a_{12}a_{22} - 8a_{22} + 3a_{22}^2 = 0 \tag{13b}$$

If the previous two equations are plotted out, one will get the same results as in Ref. 10. From Eq. (5), the squared distance

Table 1 Bound formulas of all possible perturbations for second-order systems $[|A^0| = \det(A^0) = (a_{11}^0 a_{22}^0 - a_{12}^0 a_{21}^0)]$

Perturbed elements	Bound, R_s^2
a_{11}	$R_s^2 = \min[(a_{11}^0 + a_{22}^0)^2, (\lvert A^0\rvert/a_{22}^0)^2]$
a_{12}	$R_s^2 = (\lvert A^0\rvert/a_{21}^0)^2$
a_{21}	$R_s^2 = (\lvert A^0\rvert/a_{12}^0)^2$
a_{22}	$R_s^2 = \min[(a_{11}^0 + a_{22}^0)^2, (\lvert A^0\rvert/a_{11}^0)^2]$
a_{12}, a_{22}	$R_s^2 = \min((a_{11}^0 + a_{22}^0)^2, \{\lvert A^0\rvert^2/[(a_{11}^0)^2 + (a_{21}^0)^2]\})$
a_{21}, a_{22}	$R_s^2 = \min((a_{11}^0 + a_{22}^0)^2, \{\lvert A^0\rvert^2/[(a_{11}^0)^2 + (a_{12}^0)^2]\})$
a_{11}, a_{12}	$R_s^2 = \min((a_{11}^0 + a_{22}^0)^2, \{\lvert A^0\rvert^2/[(a_{21}^0)^2 + (a_{22}^0)^2]\})$
a_{11}, a_{21}	$R_s^2 = \min((a_{11}^0 + a_{22}^0)^2, \{\lvert A^0\rvert^2/[(a_{12}^0)^2 + (a_{22}^0)^2]\})$
a_{11}, a_{22}	$R_s^2 = \min\{[(a_{11}^0 + a_{22}^0)^2/2], W_i^2\}$ $W_i = [R_{s0}Z_i/(1 - Z_i^2)]\sqrt{Z_i^2 - 4(a_{11}^0 a_{22}^0/R_{s0}^2)Z_i + 1}$ $R_{s0}^2 = [(a_{11}^0)^2 + (a_{22}^0)^2]$ and Z_i are real solutions of $a_{12}^0 a_{21}^0 Z^4 - [a_{11}^0 a_{22}^0 + 2a_{12}^0 a_{21}^0]Z^2 + R_{s0}^2 Z - \lvert A^0\rvert = 0$
a_{12}, a_{21}	$R_s^2 = \min\{W_i^2\}$ $W_i = [R_{s0}Z_i/(1 - Z_i^2)]\sqrt{Z_i^2 + 4(a_{12}^0 a_{21}^0/R_{s0}^2)Z_i + 1}$ $R_{s0}^2 = [(a_{12}^0)^2 + (a_{21}^0)^2]$ and Z_i are real solutions of $a_{11}^0 a_{22}^0 Z^4 - [a_{12}^0 a_{21}^0 + 2a_{11}^0 a_{22}^0]Z^2 - R_{s0}^2 Z + \lvert A^0\rvert = 0$
a_{11}, a_{12}, a_{21}	$R_s^2 = \min[(a_{11}^0 + a_{22}^0)^2, W_i^2]$ $W_i = [R_{s0}Z_i/(1 - Z_i^2)]\sqrt{Z_i^2 + 4(a_{11}^0 a_{22}^0/R_{s0}^2)Z_i + 1 + (Z_i^2 - 3)(a_{22}^0 Z_i/R_{s0})^2}$ $R_{s0}^2 = (a_{12}^0)^2 + (a_{21}^0)^2 + (a_{22}^0)^2$ and Z_i are real solutions of $(a_{22}^0)^2 Z^5 - a_{11}^0 a_{22}^0 Z^4 - 2(a_{22}^0)^2 Z^3 - (a_{12}^0 a_{21}^0 + 2a_{11}^0 a_{22}^0)Z^2 - R_{s0}^2 Z - \lvert A^0\rvert = 0$
a_{11}, a_{12}, a_{22}	$R_s^2 = \min\{[(a_{11}^0 + a_{22}^0)^2/2], W_i^2\}$ $W_i = [R_{s0}Z_i/(1 - Z_i^2)]\sqrt{Z_i^2 - 4(a_{11}^0 a_{22}^0/R_{s0}^2)Z_i + 1 + (Z_i^2 - 3)(a_{21}^0 Z_i/R_{s0})^2}$ $R_{s0}^2 = (a_{11}^0)^2 + (a_{21}^0)^2 + (a_{22}^0)^2$ and Z_i are real solutions of $(a_{21}^0)^2 Z^5 + a_{12}^0 a_{21}^0 Z^4 - 2(a_{21}^0)^2 Z^3 - (a_{11}^0 a_{22}^0 + 2a_{12}^0 a_{21}^0)Z^2 + R_{s0}^2 Z - \lvert A^0\rvert = 0$
a_{11}, a_{21}, a_{22}	$R_s^2 = \min\{[(a_{11}^0 + a_{22}^0)^2/2], W_i^2\}$ $W_i = [R_{s0}Z_i/(1 - Z_i^2)]\sqrt{Z_i^2 - 4(a_{11}^0 a_{22}^0/R_{s0}^2)Z_i + 1 + (Z_i^2 - 3)(a_{12}^0 Z_i/R_{s0})^2}$ $R_{s0}^2 = (a_{11}^0)^2 + (a_{12}^0)^2 + (a_{22}^0)^2$ and Z_i are real solutions of $(a_{12}^0)^2 Z^5 + a_{12}^0 a_{21}^0 Z^4 - 2(a_{12}^0)^2 Z^3 - (a_{11}^0 a_{22}^0 + 2a_{12}^0 a_{21}^0)Z^2 + R_{s0}^2 Z - \lvert A^0\rvert = 0$
a_{12}, a_{21}, a_{22}	$R_s^2 = \min\{[(a_{11}^0 + a_{22}^0)^2/2], W_i^2\}$ $W_i = [R_{s0}Z_i/(1 - Z_i^2)]\sqrt{Z_i^2 + 4(a_{11}^0 a_{22}^0/R_{s0}^2)Z_i + 1 + (Z_i^2 - 3)(a_{11}^0 Z_i/R_{s0})^2}$ $R_{s0}^2 = (a_{11}^0)^2 + (a_{12}^0)^2 + (a_{21}^0)^2$ and Z_i are real solutions of $(a_{11}^0)^2 Z^5 - a_{11}^0 a_{22}^0 Z^4 - 2(a_{11}^0)^2 Z^3 + (a_{12}^0 a_{21}^0 + 2a_{11}^0 a_{22}^0)Z^2 + R_{s0}^2 Z - \lvert A^0\rvert = 0$
$a_{11}, a_{12}, a_{21}, a_{22}$	$R_s^2 = \min\{[(a_{11}^0 + a_{22}^0)^2/2], W_i^2\}$ $W_i = [R_{s0}Z_i/(1 - Z_i^2)]\left[\sqrt{Z_i^2 - 4(\lvert A^0\rvert/R_{s0}^2)Z_i + 1}\right]$ $R_{s0}^2 = [(a_{11}^0)^2 + (a_{12}^0)^2 + (a_{21}^0)^2 + (a_{22}^0)^2]$ $Z_i = [(R_{s0}^2/\lvert A^0\rvert) \pm \sqrt{[R_{s0}^2/\lvert A^0\rvert]^2 - 4}]/2$

R_s^2 from the nominal point to any point on these two boundary surfaces is

$$R_s^2(a_{12}, a_{22}) = (a_{12} - 2)^2 + (a_{22} + 3)^2 \tag{14}$$

The two sets of equations in Eqs. (7a) and (7b) become

$$\frac{\partial W_{s1}}{\partial a_{12}} = 2(a_{12} - 2) + 3\lambda_{s1} = 0$$

$$\frac{\partial W_{s1}}{\partial a_{22}} = 2(a_{22} + 3) - 17\lambda_{s1} = 0$$

$$\frac{\partial W_{s1}}{\partial \lambda_{s1}} = -4 + 3a_{12} - 17a_{22} = 0 \tag{15a}$$

$$\frac{\partial W_{s2}}{\partial a_{12}} = 2(a_{12} - 2) + \lambda_{s2}(a_{22} - 6) = 0$$

$$\frac{\partial W_{s2}}{\partial a_{22}} = 2(a_{22} + 3) + \lambda_{s2}(-8 + a_{12} + 6a_{22}) = 0$$

$$\frac{\partial W_{s2}}{\partial \lambda_{22}} = 52 - 6a_{12} + a_{12}a_{22} - 8a_{22} + 3a_{22}^2 = 0 \tag{15b}$$

Solving Eqs. (15a) and (15b) separately gives $W_{s1,\min} = 9.426$ and $W_{s2,\min} = 51.9124$. The radius of the largest hypersphere is therefore $R_{s,\max} = 3.07$.

Table 2 **Allowable perturbations for the elements of a second-order system matrix**

Perturbed elements	μ_y	μ_p
a_{11}	1.657	3.0
a_{12}	1.657	2.0
a_{21}	0.655	1.0
a_{22}	0.396	0.667
a_{11}, a_{12}	1.0	1.414
a_{11}, a_{22}	0.382	0.4611
a_{11}, a_{21}	0.48	0.707
a_{12}, a_{21}	0.5	0.707
a_{12}, a_{22}	0.324	0.447
a_{21}, a_{22}	0.3027	0.3922
a_{11}, a_{12}, a_{21}	0.397	0.577
a_{11}, a_{12}, a_{22}	0.311	0.3597
a_{11}, a_{21}, a_{22}	0.273	0.3177
a_{12}, a_{21}, a_{22}	0.256	0.31425
a_{11}, a_{12}, a_{21}, a_{22}	0.236	0.367

Conclusion

A method of finding the largest possible hypersphere of a nominally stable linear system in the parameter space is introduced. The method can be useful to find bounds under weakly structured perturbation and highly structured perturbations. The effectiveness of the method is demonstrated by a third-order example from the literature and by finding closed-form solutions of second-order systems for all possible combinations of perturbation in the elements of the system matrix. This method is especially suited for systems of high dimensions and few perturbing variables.

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